Comparison Between Collocation Methods and Spectral Element Approach for the Stability of Periodic Delay Systems ⋆

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Abstract: This paper compares two methods that are commonly used to study the stability of delay systems. The first is a collocation technique while the second is a spectral element approach which uses the weighted residual method. Two distributions of the collocation points are compared: the first uses the extrema of Chebyshev polynomials of the first kind whereas the second uses the Legendre-Gauss-Lobatto points. The spectral element approach uses the Legendre-Gauss-Lobatto points and higher-order trial functions to discretize the delay equations while Gauss quadrature rules are used to evaluate the resulting weighted residual integrals. Two case studies are used to compare the different methods. The first case study is a $3rd$ order autonomous DDE while the second is a DDE describing the midspan deflections of an unbalanced rotating shaft with feedback gain (nonautonomous DDE). Convergence plots that compare the different rates of convergence of the described methods are also provided.

Keywords: Chebyshev collocation; delay systems; Floquet theory; Legendre collocation; spectral element; temporal finite element analysis.

1. INTRODUCTION

Delay differential equations (DDEs) have been widely used to model systems in various fields of science. Some application areas of DDEs include machining processes (Mann et al. (2005b); Butcher et al. (2009)), human balance (Stepan and Kollar (2000); Cabrera and Milton (2002)) and laser systems (Shahverdiev et al. (2009)). However, although DDEs are often necessary for correct modeling, they lead to infinite dimensional state-space which makes the associating stability analysis more difficult (Stépán (1989); Fu et al. (1989); Hale and Lunel (1993)).

Several analysis techniques have appeared in literature to investigate the stability of DDEs. For example, the stability of autonomous DDEs was studied using D-subdivision (Stépán (1989)), Continuous Time Approximation (Breda et al. (2005); Sun (2009)), and the Cluster Treatment of Characteristic Roots methods (Olgac and Sipahi (2005); Sipahi and Olgac (2006)). Moreover, the stability of autonomous as well as nonautonomous DDEs was studied using the semi-discretization method (Insperger and Stépán (2002); Insperger and Stépán (2004)), Chebyshev collocation (Butcher et al. (2004)), and temporal finite element analysis (Mann and Khasawneh (2009)).

The current work will focus on collocation and temporal finite element methods. For simplicity, only the case $T = \tau$, which appears in many applications such as machining processes (Tlusty (2000)) is considered here. In temporal finite element analysis (TFEA), the time interval of interest is discretized into a finite number of temporal elements. The method of weighted residuals is then used to transform the original DDE into a discrete map whose characteristic multipliers are analyzed to ascertain stability. The convergence in TFEA was mostly based on increasing the number of elements or h-convergence; however, recently, a spectral element version of TFEA was introduced to take advantage of spectral convergence or p-convergence in addition to h-convergence, i.e. spectral element method uses hp-convergence (Khasawneh and Mann (2010)).

On the other hand, collocation methods discretize the DDE using a set of collocation points such as Chebyshev or Legendre points. This choice of the collocation points reduces the approximation error and alleviates the Runge phenomenon (Berrut and Trefethen (2004)). A spectral differentiation matrix is then invoked to approximate the derivatives and a dynamic map is created by evaluating the DDE at the collocation points (Engelborghs et al. (2000); Butcher et al. (2004)).

The TFEA method represents a piece-wise approximation of the DDE and it inherits the flexibility of the spatial finite element methods. On the other hand, the conventional collocation methods represent a global approximation over the DDE domain. In addition, collocation methods, such

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as Chebyshev collocation, have computable uniform error bounds (Bueler (2007)), and a sketch of an *a priori* proof for the method's convergence is given in Gilsinn and Potra (2006). The general form of the state-space equations studied by the conventional Chebyshev-collocation method is

$$\dot{\mathbf{y}} = \mathbf{A}(t)\mathbf{y} + \mathbf{B}(t)\mathbf{y}(t-\tau), \tag{1}$$

where $\tau = T$ is the time delay, while $\mathbf{A}(t+T) = \mathbf{A}(t)$ and $\mathbf{B}(t+T) = \mathbf{B}(t)$ are smooth, time-periodic coefficients. This method works well and spectral convergence is obtained only when the coefficients are smooth over the entire period. The Chebyshev collocation method was also extended to delay differential equations with multiple and distributed delays (Breda et al. (2005, 2006)).

This paper compares two collocation methods (Chebyshev and Legendre) to the spectral element approach. The two collocation methods differ only in the distribution of the collocation points and the associating differentiation matrices. Whereas Chebyshev collocation uses Chebyshev-Gauss-Lobatto (LGL) points, the Legendre collocation approach uses Legendre-Gauss-Lobatto (LGL) points. Two case studies are then used to compare the results from the collocation methods to the spectral element approach.

The organization of this paper is as follows. Section 2 describes the general steps of the collocation approach without specifying the specific choice of the collocation nodes. The Chebyshev and Legendre distributions of collocation points are shown is sections 2.1 and 2.2, respectively. Section 3 describes briefly the main steps of the spectral element approach. Section 4 shows two case studies and the comparison plots while Section 5 provides a conclusion from the results.

2. STABILITY ANALYSIS OF DDES USING COLLOCATION METHODS

Collocation methods are used to study the stability of DDEs with smooth coefficients such as (1). Therefore, a global approximation is justified and the residual error is guaranteed to vanish as the approximation order is increased. In the conventional collocation methods, the procedure involves choosing a suitable set of collocation points and obtaining the corresponding spectral differentiation matrix.

In the collocation methods that are based on Chebyshev and Legendre basis the definition domain of the collocation points is usually the interval $[-1, 1]$. The shift from the definition domain to an arbitrary domain $[a, b]$ is given by

$$\tilde{u} = \frac{b-a}{2}u + \frac{b+a}{2}, \tag{2}$$

where $u \in [-1, 1]$ and $\tilde{u} \in [a, b]$. The specific computations of the collocation points and the generation of the spectral differentiation matrices is discussed in Sections 2.1 and 2.2.

For example, let the order of the state-space equation (1) be q. Using a Chebyshev discretization scheme and evaluating (1) at the collocation points on the interval $[0, \tau]$ gives

$$\hat{\mathbf{D}}\mathbf{m}_{r+1} = \hat{\mathbf{M}}_A\mathbf{m}_{r+1} + \hat{\mathbf{M}}_B\mathbf{m}_r, \tag{3}$$

where the $q \times 1$ vector $\mathbf{m}_{r+1}$ contains the values of $\mathbf{y}(t)$ in the interval $t \in [0, \tau]$ while the $q \times 1$ vector $\mathbf{m}_r$ contains the values of the initial function $\boldsymbol{\psi}(t) = \mathbf{y}(t-\tau)$ in $t \in [-\tau, 0]$ according to

$$\mathbf{m}_{r+1} = \begin{bmatrix} \mathbf{y}(t_0) \\ \mathbf{y}(t_1) \\ \vdots \\ \mathbf{y}(t_N) \end{bmatrix}, \quad \mathbf{m}_r = \begin{bmatrix} \boldsymbol{\psi}(t_0) \\ \boldsymbol{\psi}(t_1) \\ \vdots \\ \boldsymbol{\psi}(t_N) \end{bmatrix}. \tag{4}$$

In order to describe the matrix $\hat{\mathbf{D}}$, the $q(N+1)$ square differential operator $\mathbb{D}$ is defined first as the Kronecker product

$$\mathbb{D} = \mathbf{D} \otimes \mathbf{I_q} \tag{5}$$

where $\mathbf{D}$ depends on the collocation scheme used and its entries will be specified in Sections 2.1 and 2.2, and $\mathbf{I}_q$ is an identity matrix. The matrix $\hat{\mathbf{D}}$ is then a modified version of the spectral differentiation matrix $\mathbb{D}$ with the following changes (1) multiplying by $2/\tau$ to account for the shift $[-1, 1] \to [0, \tau]$, and (2) changing the last q rows to $[\mathbf{0}_q\ \mathbf{0}_q \ldots \mathbf{I}_q]$ where $\mathbf{0}_q$ and $\mathbf{I}_q$ are $q \times q$ null and identity matrices, respectively. The change in the last q rows of $\hat{\mathbf{D}}$, (and the last q rows of $\hat{\mathbf{M}}_A$ and $\hat{\mathbf{M}}_B$ as will be shown shortly), is necessary to enforce the periodicity condition which, for this particular case, implies that the states are equal at the end of one period and the beginning of the subsequent one.

The $q(N+1)$ square matrix $\hat{\mathbf{M}}_A$ has the entries

$$\hat{\mathbf{M}}_A = \begin{bmatrix} \mathbf{A}(t_0) & & & & \\ & \mathbf{A}(t_1) & & & \\ & & \ddots & & \\ & & & \mathbf{A}(t_{N-1}) & \\ \mathbf{0}_q & \mathbf{0}_q & \cdots & \mathbf{0}_q & \mathbf{0}_q \end{bmatrix}, \tag{6}$$

where $\mathbf{A}(t_i)$ is the value of $A(t)$ in (1) evaluated at the ith collocation point. Similarly, the square $q(N+1)$ matrix, $\hat{\mathbf{M}}_B$, has the entries

$$\hat{\mathbf{M}}_B = \begin{bmatrix} \mathbf{B}(t_0) & & & & \\ & \mathbf{B}(t_1) & & & \\ & & \ddots & & \\ & & & \mathbf{B}(t_{N-1}) & \\ \mathbf{I}_q & \mathbf{0}_q & \cdots & \mathbf{0}_q & \mathbf{0}_q \end{bmatrix}, \tag{7}$$

where $\mathbf{B}(t_i)$ is the value of $B(t)$ in (1) evaluated at the ith collocation point.

Equation (3) can be rearranged to obtain the dynamic map

$$\mathbf{m}_{r+1} = \mathbf{U}\mathbf{m}_r, \tag{8}$$

where the monodromy matrix $\mathbf{U} = (\hat{\mathbf{D}} - \hat{\mathbf{M}}_A)^{-1}\hat{\mathbf{M}}_B$ is the finite approximation to the infinite-dimensional monodromy operator. The stability of the system described by (1) can then be determined by examining the eigenvalues of $\mathbf{U}$ using Floquet theory. The asymptotic stability criteria states that the system is stable if all the characteristic multipliers, or eigenvalues, of the monodromy operator are within the unit circle in the complex plane. Alternatively, the inversion of $(\hat{\mathbf{D}} - \hat{\mathbf{M}}_A)$ can be avoided by setting the determinant $|\hat{\mathbf{M}}_B - \mu(\hat{\mathbf{D}} - \hat{\mathbf{M}}_A)|$ to zero, where μ is the characteristic multiplier.

2.1 Chebyshev collocation nodes

The $N+1$ CGL collocation points are the extrema of the Nth order Chebyshev function, and they are found from

$$z_n = \cos\frac{n\pi}{N} \qquad \text{where } n = 0, 1, \ldots, N, \tag{9}$$

where the change of variables in (2) can be used to shift these points from $[-1, 1]$ to any other interval. Evaluating the DDE at the Chebyshev points minimizes the approximation error and allows using the simplified expression for the Chebyshev spectral differentiation matrix (Sinha and Butcher (1996)). The entries of the Chebyshev differentiation matrix are given by

$$D_{00} = -D_{NN} = \frac{2N^2+1}{6}, \tag{10a}$$

$$D_{kk} = \frac{-z_k}{2(1-z_k^2)}, \ \ k = 1, \ldots, N-1 \tag{10b}$$

$$D_{kn} = \frac{-\bar{c}_k(-1)^{k+n}}{\bar{c}_n(z_k - z_n)}, \ \ k \neq n, \ \ k, n = 0, \ldots, N, \tag{10c}$$

where $\bar{c}_{[\cdot]} = 2$ if $[\cdot] = 0$ or $[\cdot] = N$, and is 1 otherwise. Equation (10) gives the expression for the matrix $\mathbf{D}$ used in (5) when using a Chebyshev collocation scheme.

2.2 Legendre collocation nodes

Similar to the Chebyshev points, The LGL points also offer a set of well-distributed points suitable for collocation methods (Bloom et al. (1992)). The $N+1$ LGL points are the roots of the polynomial $(1-u^2)L'_N(u)$ where u ranges from -1 to 1 and $L_N(u)$ is the Legendre polynomial of order N (Vu and Deeks (2006)). These points can be shifted to any arbitrary interval through the relation given in (2).

Using the LGL nodes gives a simplified expression for the entries of $\mathbf{D}$ used in (5) when a Legendre collocation approach is used. Specifically, the dimension of this matrix is $(N+1) \times (N+1)$ and its entries are given by (Elnagar et al. (1995))

$$D_{00} = -D_{NN} = -\frac{N(N+1)}{4}, \tag{11a}$$

$$D_{km} = \begin{cases} \dfrac{L_N(t_k)}{L_N(t_m)} \dfrac{1}{(t_k - t_m)}, & k \neq m \\ 0, & \text{otherwise.} \end{cases} \tag{11b}$$

Upon obtaining the collocation points and the corresponding differentiation matrices, the steps outlined in Section 2 can be followed to ascertain the stability of the DDE using a Legendre collocation scheme.

3. SPECTRAL ELEMENT APPROACH

The spectral element method for DDEs introduced in Khasawneh and Mann (2010) is an improvement over the state-space TFEA approach presented in Mann and Khasawneh (2009). The advantage of the spectral element approach over the state-space TFEA method is the ability to achieve an *hp*-convergence scheme. This allowed taking full advantage of spectral convergence, which was lacking in the conventional approach, while maintaining the flexibility associated with finite element methods. While a more detailed description of the method is presented in Khasawneh and Mann (2010), only the basic steps are described in the present paper due to space limitations.

In TFEA, the time interval of interest is discretized into a finite number of temporal elements. Then, using the method of weighted residuals, the original DDE is transformed into the form of a discrete map whose characteristic multipliers, i.e. eigenvalues, are analyzed to ascertain stability. For example, Fig. 1 shows a possible discretization of the timeline in TFEA using a total of 4 elements: 2 in the current period and 2 in the delayed period.

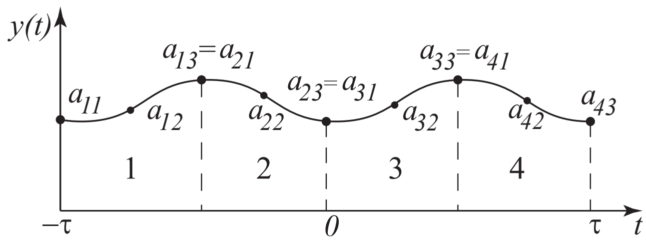

Fig. 1. Timeline for the state vector, $\mathbf{y}$, over a time interval of 2τ. Dots denote the locations where the coefficients of the assumed solution are equivalent to the state variables. The beginning and end of each temporal element is marked with dotted lines.

For example, consider a *qth* order DDE with time-periodic coefficients of the form shown in (1). Since the coefficients are time-periodic, the discretization needs to ensure that these terms take on the correct values within each element. To elaborate, assume that the time duration of each element is t_j, the discretization then requires the substitution $t = \sigma + (j-1)t_j = (\eta + j - 1)t_j$ into $\mathbf{A}(t)$ and $\mathbf{B}(t)$ in (1) to ensure that they assume the correct values over the entire period. The expressions for the current and the delayed state variables are then written as the vectors

$$\mathbf{x}_j(t) = \sum_{i=1}^{n+1} \mathbf{a}_{ji}\phi_i(\eta), \tag{12a}$$

$$\mathbf{x}_j(t-\tau) = \sum_{i=1}^{n+1} \mathbf{a}_{j-E,i}\phi_i(\eta). \tag{12b}$$

where $\eta = \sigma/t_j$ is the normalization of the local coordinate σ with respect to t_j, $n+1$ is the number of the polynomial trial functions $\phi(\eta)$ used, and E is the number of elements in each period. In contrast to the Hermite trial functions which were used in the conventional TFEA, the spectral element approach uses Lagrange trial functions calculated from the barycentric formula on the LGL points (Berrut and Trefethen (2004),Higham (2004)). The equation for obtaining these trial functions uses the Lagrange barycentric formula and it is given by (Berrut and Trefethen (2004))

$$\phi_i(\eta) = \frac{\frac{\varpi_i}{\eta - \eta_i}}{\sum_{k=1}^{n+1} \frac{\varpi_k}{\eta - \eta_k}}, \tag{13}$$

where ϖ_k are the barycentric weights given by

$$\varpi_k = \frac{1}{\prod_{k \neq j}(\eta_j - \eta_k)}, \qquad j = 1, \ldots, n+1. \tag{14}$$

The barycentric formula requires less computational effort and has better numerical stability than the conventional Lagrange representation (see Berrut and Trefethen (2004); Higham (2004)); therefore, it was used to generate the trial functions in the present study.

In addition to being a more efficient tool to generate the trial functions, the barycentric weights can be used to obtain the value of the derivative of the trial functions evaluated at the interpolation nodes according to

$$\phi_i'(\eta_k) = \frac{\varpi_i/\varpi_k}{\eta_i - \eta_k}. \quad (15)$$

These values are useful in evaluating the weighted residual integrals using a Legendre-Gauss-Lobatto quadrature.

In fact, another distinguishing feature of the spectral element approach is that it employs a LGL quadrature rule, as opposed to analytical integrations, to evaluate the weighted residual terms. In addition to handling more complex integrals, the evaluation of these integrals is reduced to a weighted summation of matrices. Furthermore, choosing the quadrature points to be identical to the interpolation points allows the properties of Lagrange interpolants to simplify the quadrature even further. These features yield a spectral element method that does not require obtaining the expressions of the trial functions, only the barycentric weights associated with them. These weights can be obtained only by knowing the set of interpolation points within each element. In other words, with the spectral element approach it becomes possible to evaluate the weighted residual integrals using only the information from the temporal mesh and the state-space matrices.

The approximate expressions of (12) are then substituted into (1) which yields a discretized version of the DDE in addition to an error term due to the approximation procedure. The error can be minimized by multiplying by a set of independent test functions and setting the integral over the duration of the element to zero (Reddy (1993)). This is called the method of weighted residuals which also generates a sufficient number of equations to create the dynamic map.

Applying each test function in turn, the resulting equations for each element can be combined into a global matrix equation. A dynamic map can then be constructed similar to (8) and the stability of the system can be determined from the eigenvalues of the map.

4. CASE STUDIES

To compare the collocation methods to the spectral element approach, the stability of two DDEs was investigated. The first example studies the stability of an autonomous $3rd$ order DDE whereas the second example investigates the stability of an unbalanced rotating shaft with feedback gain; which is described by coupled second order DDEs with time-periodic coefficients. In addition, the convergence of the spectral radius, at a point near the stability boundaries, is compared for all methods. Although the spectral element approach described in this paper can use an *hp*-convergence scheme, in this study the number of elements was fixed to 1 and only the order of the trial functions (i.e. p-convergence) was used to contrast the results with those of the collocation methods. For the collocation methods, the number of collocation points was increased to obtain spectral convergence.

The reference $\mu*$ for comparing the spectral radius for all methods was the magnitude of the maximum eigenvalue obtained from one of the collocation methods for a high number of collocation points $N = N_{\max}$, i.e. $\mu* = \max(|\mu(N_{\max}|)$. This is due to the well known spectral rate of convergence of collocation methods. Specifically, the absolute difference $|\lambda - \mu*|$, where $\lambda = \max|\mu|$ is the magnitude of the maximum eigenvalue obtained from one of the methods, was plotted either as a function of the order of the Lagrange trial functions or of the number of collocation points. This provided a common reference for comparing the rates of convergence of the different methods. The common reference for the first example was obtained from the Chebyshev collocation approach whereas for the second example Legendre collocation provided the basis of comparison. In both examples, the maximum number of collocation points as well as the maximum order of Lagrange trial functions was set to 40. A result on the *a posteriori* error bounds that can be used to choose the number of collocation points can be found in Ref. Bueler (2007).

4.1 Third order example

Determining the stability of a third order delay equation would not have been possible with the original version of TFEA (see Bayly et al. (2003); Mann et al. (2004); Mann and Young (2006); Mann et al. (2005a)). Thus, we consider the following third order delay equation

$$\frac{d^3x}{dt^3} + \alpha\dot{x} + \beta x(t-\tau) = 0\,, \quad (16)$$

with a delay $\tau = 1$ and control parameters α and β.

The stability diagram and the convergence plots for the (α, β) point $(202.04, 28.38)$ are shown in Fig. 2. Figure 2-(b) shows that all three methods showed a spectral (exponential) rate of convergence. However, the Chebyshev method converged slightly faster than the Legendre method. In addition, the spectral element approach converged faster than the collocation methods.

Fig. 2. The stability diagram for (16) and the spectral convergence plots for the point $(202.04, 28.38)$ in the parameter space. In graph (a), the stable regions are shaded whereas the unstable regions are left unshaded. Graph (b) shows the convergence of the maximum eigenvalue as a function of (1) the number of Legendre collocation points (triangles), (2) the number of Chebyshev collocation points (squares), and (3) the order of the Lagrange trial functions (dots).

4.2 A non-autonomous, time-periodic system

The spectral element approach can also be used to study the stability of non-autonomous systems with time-periodic coefficients. For example, consider the system of coupled DDEs given by Yamamoto and Ishida (2001)

$$\ddot{x} + 2\zeta\,\dot{x} + (1 - \delta\cos 2\Omega t)x - \delta\sin(2\Omega t)\,y = bx(t-T), \quad (17a)$$

$$\ddot{y} + 2\zeta\,\dot{y} + (1 + \delta\cos 2\Omega t)y - \delta\sin(2\Omega t)\,x = by(t-T). \quad (17b)$$

These equations describe the non-dimensionalized deflections of the midspan of an asymmetric shaft where ζ is damping, δ is a non-dimensional parameter describing the amount of asymmetry in the system, b is the non-dimensional feedback gain, Ω is the non-dimensional rotational frequency, and $T = \pi/\Omega$ is a constant delay. The delayed term is non-instantaneous or delayed feedback that is used to stabilize the shaft. Equation (17) can be written in state-space form to match the form of (1) and the stability analysis described in the previous sections can then be used.

A stability diagram for (17) is shown in Fig. 3. In this figure, the feedback gain was assigned the value $b = 0.04$ and the unstable region was shaded. Figure 3-(b) shows the convergence plots for the magnitude of the maximum Floquet multiplier λ at the (Ω, δ) point $(1.16, 0.3)$ for the different methods. Specifically, the convergence is shown as a function of three quantities: (1) the number of points in Legendre collocation (triangles), (2) the number of Chebyshev collocation points (squares), and (3) the order of the trial functions in the spectral element approach (dots).

Similar to the previous example, it can be seen that the spectral element approach and the collocation methods had a spectral rate of convergence. Whereas the Chebyshev method converged slightly faster than Legendre, the spectral element approach converged faster than both collocation methods.

Fig. 3. A stability diagram for Eq. (17) and the spectral convergence plots for the (Ω, δ) point $(1.16, 0.3)$ in the parameter space. The parameters used in graph (a) were $\zeta = 0.02$ and $b = 0.04$. The unstable regions are shaded whereas the stable regions are left unshaded. Graph (b) shows the convergence of the maximum eigenvalue as a function of (1) the number of Legendre collocation points (triangles), (2) the number of Chebyshev collocation points (squares), and (3) the order of the Lagrange trial functions (dots).

5. CONCLUSIONS

This paper provided a comparison between two methods for the stability determination of delay systems. Namely, a collocation based method and a spectral element approach, where the latter is based on the weighted residual method, were used to obtain the stability of two DDEs. For the collocation method, two sets of collocation nodes were used: Legendre-Gauss-Lobatto points and Chebyshev-Gauss-Lobatto points. Two case studies were used to compare the different methods. The first case study was a 3*rd* order DDE while the second case was a system of coupled, second-order DDEs with nonautonomous coefficients. Although the converged stability diagrams obtained from the different method were almost identical, the spectral convergence plots for points near the stability boundaries showed some differences between the various methods.

For the cases considered here, it was found that the spectral element approach converged faster than both Chebyshev collocation and Legendre collocation. In addition, the Chebyshev collocation rate of convergence was slightly faster than its Legendre counterpart. Therefore, the spectral element approach, which uses higher-order trial functions and a Gauss quadrature rule to obtain the weighted residual integrals, was found to be an accurate and efficient approach for the stability studies of DDEs.

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